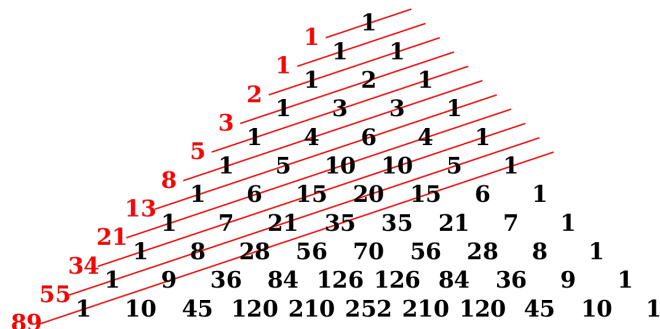# Grade 9/10 Math Circles
February 8, 2023
The Shape of You - Problem Set

Pascal's Triangle

1. Prove that the sum of the entries in the n-th row of Pascal's Triangle equals 2^{n-1}. Isn't this a little weird?

2. If you look at Pascal's Triangle for long enough, you might notice this pattern:
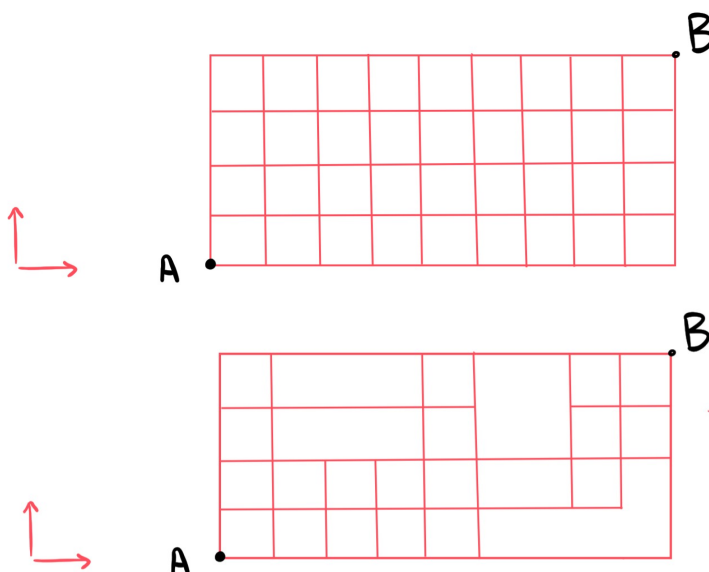

   Give an argument for why the Fibonacci numbers magically appear here as the sums of the diagonals in Pascal's Triangle...

Path Counting

3. Count (and then defend your answer) the number of ways to get from point A to point B in the following diagrams. The only allowed directions of motions are indicated by the red arrows.

Combinations and Permutations

4. Suppose there are five friends who want to organize a committee of three people[1]. Compute the number of way to do this, ie. the number of ways to choose 3 people (or objects) out of a set of 5. Note: it may be helpful to draw a 5-simplex.

5. Suppose now that you are choosing a 3-letter long password out of the 5 letters A,B,C,D,E, and you cannot use the same letter twice in the password (just like you could not have one person count as two people in the previous question). Compute the total number of possible passwords.

6. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

[1]Why these people spend their time forming committees is beyond me.

"Word" Problems

7. How many seven letter words are possible using the letters S, I, M, P, L, E, X? Define a word to be *any sequence of letters*. Hint: try calculating this for a smaller number of letters and look for a pattern. What if you had n letters instead of 7?

8. How many eight letter words are possible using the letters W, A, T, E, R, L, O, O? Note that two words are the same if they look like the same sequence of letters.

9. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

10. How many five letter words are possible using the letters N, N, C, C, C?

11. Compare your answer above to your answer in the (rather absurd) committee problem in question 4. Did you get the same answer? Different answers? Why?

    Hint: I *coincidentally* always represent "chosen" by the letter C and represent "not chosen" by the letter N. Hm.

12. How many five letter words are possible using the letters L, L, R, R, R?

13. Count the number of ways to get from point A to point B in the following diagrams. The only allowed directions of motions are indicated by arrows.
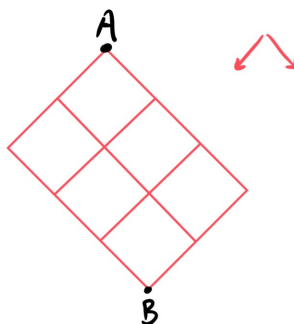
14. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

    Hint: It *just so happens* that I always represent "left" by the letter L and represent "right" by the letter R. Hmmm.

15. **Extra super hardcore brainpower challenge**: Use the above exercises to find and *prove* a closed form formula for $\binom{n}{k}$.

Closed form means that the formula should look like some arithmetic operations between numbers and the variables n and k. *Prove* means give an airtight argument that your most annoying friend wouldn't be able to argue with.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is called a *recursive* formula because you need to recompute earlier smaller values and combine them before you get $\binom{n}{k}$. In particular, it isn't *closed form*.